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# Note

## On the Advantages of the Vorticity–Velocity Formulation of the Equations of Fluid Dynamics*

Two distinctly different approaches have been utilized in the literature for the numerical solution of the equations of viscous flow in three dimensions. In the more common approach, the momentum equation, which contains both the velocity and pressure, is solved numerically along with a derived Poisson equation for the pressure (i.e., the pressure–velocity or primitive variable formulation [1–3]). The alternative approach is based on eliminating the pressure from the momentum equation by the application of the curl. In this manner, a vorticity transport equation is solved numerically in lieu of the momentum equation (i.e., the vorticity–velocity formulation [4–6]). The purpose of the present note is to explore in more detail the properties of these disparate numerical approaches. It will be shown that the vorticity–velocity formulation has a striking advantage when applied to problems in non-inertial frames of reference. More specifically, there exists an intrinsic vorticity–velocity formulation wherein all non-inertial effects (arising from both the rotation and translation of the frame of reference relative to an inertial framing) *only enter into the solution of the problem through the implementation of initial and boundary conditions*. This is in stark contrast to the pressure–velocity formulation, where non-inertial effects appear directly in the momentum equation in the form of Coriolis and Eulerian accelerations—a state of affairs which can give rise to a variety of numerical problems [2]. A detailed exposition of this interesting property of the vorticity–velocity formulation will be presented along with a brief discussion of other advantages of this approach.

For simplicity, we will restrict our attention to the analysis of viscous incompressible flow governed by the Navier–Stokes equation and continuity equation which, respectively, take the form

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \nu \nabla^2 \mathbf{v}, \tag{1}$$

$$\nabla \cdot \mathbf{v} = 0, \tag{2}$$

where $\mathbf{v}$ is the velocity vector, $p$ is the pressure, and $\nu$ is the kinematic viscosity of

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the fluid. Here, the validity of (1) requires that the external body forces be conservative and that the frame of reference be inertial. In an *arbitrary* non-inertial frame of reference (which can rotate with a time-dependent angular velocity $\mathbf{\Omega}(t)$ and translate with a time-dependent velocity $\mathbf{V}_0(t)$ relative to the origin 0 of an inertial framing), the Navier–Stokes equation takes the more complex form [7]

$$\frac{\partial \mathbf{v}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{v}+\dot{\mathbf{\Omega}}\times\mathbf{r}+\mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{r})+\dot{\mathbf{V}}_0+2\mathbf{\Omega}\times\mathbf{v}=-\nabla p+\nu\nabla^2\mathbf{v}. \tag{3}$$

Here, $\mathbf{r}$ is the position vector and the non-inertial terms on the left-hand side of (3) are, respectively, referred to as the Eulerian, centrifugal, translational, and Coriolis accelerations. The continuity equation still assumes the same form (2) in any non-inertial frame of reference.

By the introduction of a modified pressure $P$ which includes the centrifugal and translational acceleration potentials, the non-inertial form of the Navier–Stokes equation (3) can be simplified considerably. More specifically, (3) can be written in the equivalent form

$$\frac{\partial \mathbf{v}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{v}+\dot{\mathbf{\Omega}}\times\mathbf{r}+2\mathbf{\Omega}\times\mathbf{v}=-\nabla P+\nu\nabla^2\mathbf{v}, \tag{4}$$

where

$$P=p+\tfrac{1}{2}(\mathbf{\Omega}\cdot\mathbf{r})^2-\tfrac{1}{2}\Omega^2r^2+\dot{\mathbf{V}}_0\cdot\mathbf{r}. \tag{5}$$

In the pressure–velocity formulation, Eq. (4) is solved in conjunction with a Poisson equation for the pressure which is obtained by taking the divergence of (4). Hence, the governing equations to be solved numerically in this approach can be summarized as follows,

$$\frac{\partial \mathbf{v}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{v}+\dot{\mathbf{\Omega}}\times\mathbf{r}+2\mathbf{\Omega}\times\mathbf{v}=-\nabla P+\nu\nabla^2\mathbf{v}, \tag{6}$$

$$\nabla^2 P=-\mathrm{tr}(\nabla\mathbf{v}\cdot\nabla\mathbf{v})+2\mathbf{\Omega}\cdot\boldsymbol{\omega}, \tag{7}$$

subject to the initial and boundary conditions

$$\mathbf{v}=\mathbf{v}_0 \qquad \text{at } t=t_0, \tag{8}$$

$$\left.\begin{aligned}\mathbf{v}&=\mathbf{v}_B\\ P&=P_B\end{aligned}\right\} \qquad \text{on } B. \tag{9}$$

In (7) and (9), $\mathrm{tr}(\cdot)$ denotes the trace, $\boldsymbol{\omega}$ is the vorticity vector, and $B$ denotes the boundary surface of the region. Of course, Eqs. (6) and (7) must be solved subject to the continuity equation (2). Hence, the solution for the velocity $\mathbf{v}$ must be projected in some suitable fashion onto the space of solenoidal vectors (of course, the continuity equation can be satisfied identically by the use of a stream function for plane and axisymmetric flows or by the use of a vector potential for more general flows).

It is quite clear that the forms of (6) and (7) (and, hence, their mathematical character) change depending on whether or not the frame of reference is inertial. Consequently, a particular numerical algorithm which works well for a given class of flows in an inertial frame of reference may not do so for the same class of flows in a non-inertial framing.[1] It will now be demonstrated that the vorticity–velocity formulation does not suffer from this deficiency.

The vorticity–velocity formulation is based on the vorticity transport equation which is obtained by taking the curl of (4). This equation takes the form

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \mathbf{v} \cdot \nabla \boldsymbol{\omega} = \boldsymbol{\omega} \cdot \nabla \mathbf{v} + \nu \nabla^2 \boldsymbol{\omega} + 2\boldsymbol{\Omega} \cdot \nabla \mathbf{v} - 2\dot{\boldsymbol{\Omega}} \tag{10}$$

in any non-inertial frame of reference, where

$$\boldsymbol{\omega} = \nabla \times \mathbf{v} \tag{11}$$

is the vorticity vector. It is clear that the velocity and vorticity are also connected through the Poisson equation

$$\nabla^2 \mathbf{v} = -\nabla \times \boldsymbol{\omega}, \tag{12}$$

which is a direct consequence of the vector identity

$$\nabla \times (\nabla \times \mathbf{v}) = \nabla(\nabla \cdot \mathbf{v}) - \nabla^2 \mathbf{v}. \tag{13}$$

The intrinsic vorticity $\mathbf{W}$, defined by

$$\mathbf{W} = \boldsymbol{\omega} + 2\boldsymbol{\Omega}, \tag{14}$$

can be introduced which represents the vorticity relative to an inertial frame of reference. Since $\boldsymbol{\Omega}$ is spatially homogeneous (i.e., $\nabla \boldsymbol{\Omega} = 0$), it is a simple matter to show that the non-inertial form of the vorticity–velocity formulation can be written as follows:

$$\frac{\partial \mathbf{W}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{W} = \mathbf{W} \cdot \nabla \mathbf{v} + \nu \nabla^2 \mathbf{W}, \tag{15}$$

$$\nabla^2 \mathbf{v} = -\nabla \times \mathbf{W}. \tag{16}$$

[1] For example, two-level explicit finite difference schemes such as upwind differencing which have been used successfully in the description of two-dimensional viscous flows in an inertial framing are unstable in rotating frames due to the Coriolis terms (for stability, three-level schemes should be used where the Coriolis terms are centered in time; see Williams [2]).

Equations (15)–(16) must be solved (in some region $R$ with a boundary surface $B$) subject to the initial and boundary conditions

$$\mathbf{W} = (\nabla \times \mathbf{v})_0 + 2\mathbf{\Omega}_0 \qquad \text{at } t = t_0, \tag{17}$$

$$\left.\begin{aligned} \mathbf{v} &= \mathbf{v}_B \\ \mathbf{W} &= (\nabla \times \mathbf{v})_B + 2\mathbf{\Omega} \end{aligned}\right\} \qquad \text{on } B. \tag{18}$$

Of course, it is well known that the vorticity, as well as the intrinsic vorticity, are solenoidal, i.e.,

$$\nabla \cdot \mathbf{W} = 0, \tag{19}$$

and, hence, the solutions for $\mathbf{W}$ and $\mathbf{v}$ must, in some suitable fashion, he projected onto the space of solenoidal vectors.

This vorticity–velocity formulation of fluid dynamics represented by Eqs. (15)–(18) has the striking property that *non-inertial effects only enter into the solution of the problem through the implementation of initial and boundary conditions.* Consequently, the basic structure of the numerical algorithm (i.e., the numerical formulation of (15)–(16)) will be independent of whether or not the frame of reference is inertial—a situation which greatly enhances the general applicability of any Navier–Stokes computer code which is developed based on this approach.

At this point, a few comments should be made concerning the alternate ways in which the velocity field can be calculated in the vorticity–velocity formulation. Instead of solving the Poisson equation (16), it is possible to solve the defining equation for vorticity directly, i.e.,

$$\nabla \times \mathbf{v} = \boldsymbol{\omega} = \mathbf{W} - 2\mathbf{\Omega} \tag{20}$$

(see Gatski, Grosch, and Rose [6, 8]). Of course, for plane or axisymmetric flows, there exists a stream function $\psi$ such that [7]

$$\mathbf{v} = \boldsymbol{\lambda} \times \nabla\psi, \tag{21}$$

$$\nabla \times (\boldsymbol{\lambda} \times \nabla\psi) = \mathbf{W} - 2\mathbf{\Omega}, \tag{22}$$

where $\boldsymbol{\lambda} = \nabla\chi$ and $\chi$ is the coordinate that the flow is independent of (for plane flows, (22) reduces to the Poisson equation $\nabla^2\psi = W - 2\Omega$). While the motion of the frame of reference *does* enter into the equations of motion in these alternate vorticity–velocity formulations, it does so in a much less significant way than in the pressure–velocity formulation. To be specific, the transport equation which is solved (i.e., Eq. (15)) does not contain any frame-dependent terms and, at each time step, the partial differential equation for the determination of the velocity field is only altered by the addition of a *constant* forcing function in the form of $2\mathbf{\Omega}$ (the added term on the right-hand side of (20) and (22)).

Finally, it would be of value to mention some other advantages of the vorticity–velocity formulation. More difficulties have been known to arise in the implementation of pressure boundary conditions than vorticity boundary conditions [1, 2] (of course, both boundary conditions must usually be derived). Difficulties in satisfying the continuity equation in the pressure–velocity formulation have also been known to give rise to numerical instabilities [1]. Furthermore, in the vorticity–velocity approach, the vorticity vector is calculated directly. This is of considerable value since the vorticity field can play an important role in characterizing certain features of turbulence [9]. While it is certainly not being suggested that the pressure–velocity formulation be abandoned, this study does indicate that the vorticity–velocity formulation can have distinct advantages when applied to an important class of viscous flows.

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